

11/12

Last Time: Green's Theorem.

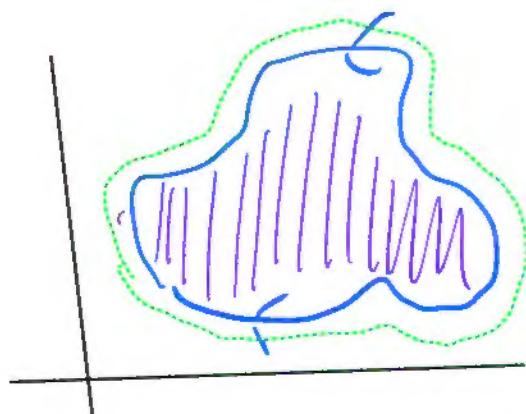
Prop (Green's Theorem): Suppose

D is a region in the plane with its boundary a piecewise smooth simple-closed curve. If $P(x,y)$

and $Q(x,y)$ have cts

partial derivatives on some open region, " R ", containing D , then

$$\left[\int_{\partial D} P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \right]$$

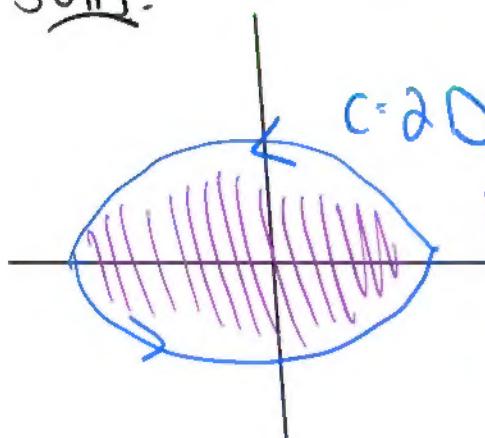


Ex: Compute $\int_C y^4 dx + 2xy^2 dy$ for

C the positively oriented ellipse

$$x^2 + 2y^2 = 2.$$

Soln:



Notice:

like a circle. So
make a substitution
to make it curve like
then use polar coords.
(Transformation)

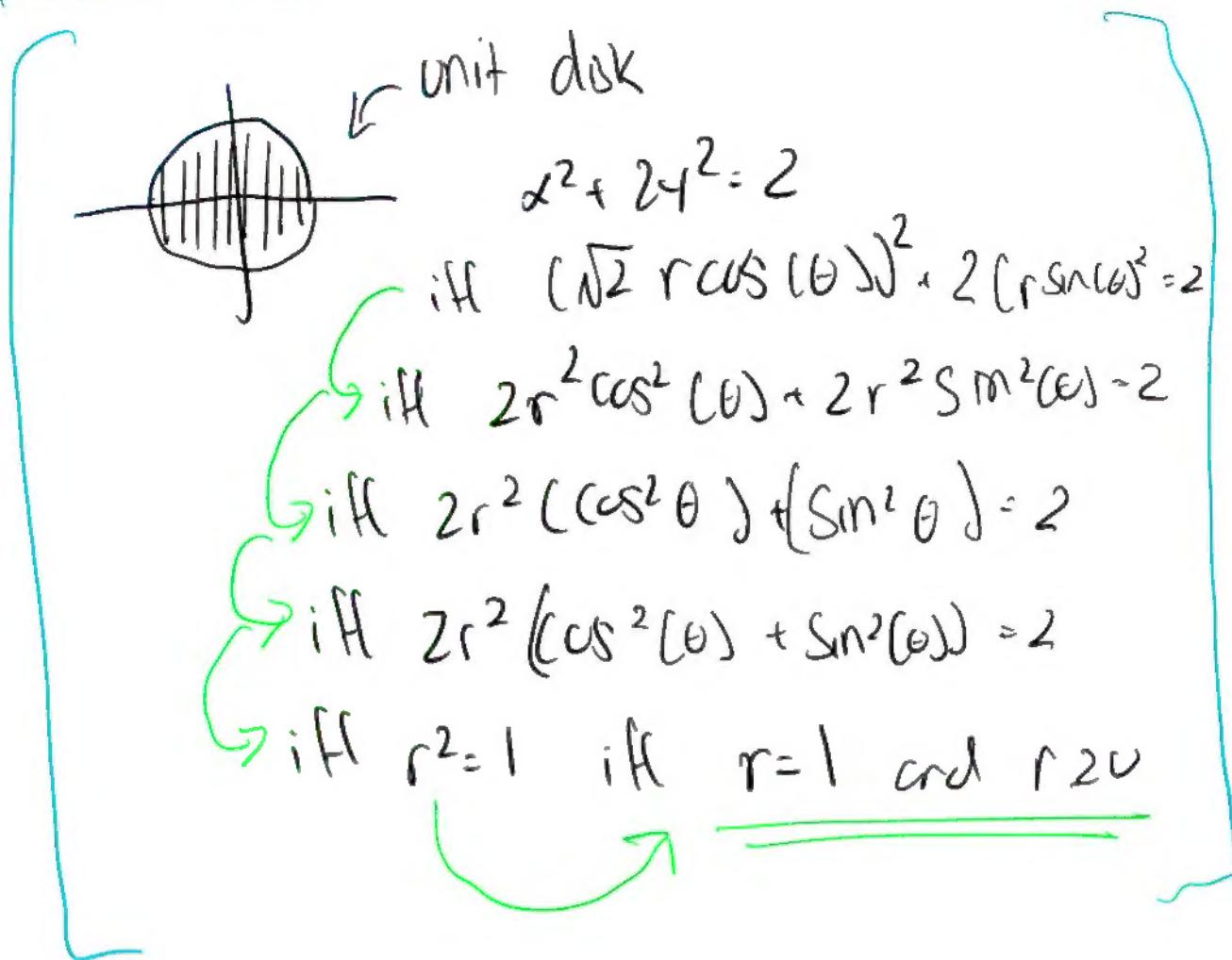
$$\begin{cases} x = \sqrt{2} r \cos \theta \\ y = r \sin \theta \end{cases}$$

By Green's Theorem: * $0 \leq r \leq 1$ *
 $0 \leq \theta \leq 2\pi$

$$\int_C y^4 dx + 2xy^2 dy$$

$$\begin{aligned} &= \iint_D \left(\frac{\partial}{\partial x} [2xy^2] - \frac{\partial}{\partial y} [y^4] \right) dA \\ &= \iint_D (2y^2 - 4y^3) dA \end{aligned}$$

Notice cont.



Now compute Jacobian:

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$\begin{aligned}
 &= \det \begin{bmatrix} \sqrt{2} \cos \theta & -\sqrt{2} r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \\
 &= \sqrt{2} r \cos^2 \theta + \sqrt{2} r \sin^2 \theta
 \end{aligned}$$

$$= \sqrt{2} r$$

$$\int_C y^4 dx + 2xy^2 dy = \iint_D (2y^2 - 4y^3) dA$$

$$\int_{r=0}^1 \int_{\theta=0}^{2\pi} (2(r \sin \theta))^2 - 4(r \sin \theta)^3 \sqrt{2} r \, d\theta dr$$

$$\int_{r=0}^1 \int_{\theta=0}^{2\pi} 2\sqrt{2} r^3 (\sin^2 \theta - 2r \sin^3 \theta) \, d\theta dr$$

$$\int_{r=0}^1 2\sqrt{2} r^3 \int_{\theta=0}^{2\pi} \sin^2 \theta (1 - 2r \sin(\theta)) \, d\theta dr$$

$$= \int_{r=0}^1 2\sqrt{2} r^3 \int_{\theta=0}^{2\pi} (1 - \cos^2 \theta) (1 - 2r \sin \theta) d\theta dr$$

eval. inner integral:

$$\int_{\theta=0}^{2\pi} (1 - \cos^2 \theta) (1 - 2r \sin \theta) d\theta$$

$$= \int_{\theta=0}^{2\pi} (1 - \cos^2 \theta) d\theta - 2r \int_{\theta=0}^{2\pi} (1 - \cos^2 \theta) \sin \theta d\theta \quad \begin{aligned} u &= \cos \theta \\ du &= -\sin \theta d\theta \end{aligned}$$

$$= \int_{\theta=0}^{2\pi} \left(1 - \frac{1}{2}(1 + \cos 2\theta)\right) d\theta - 2r \int_{\theta=0}^{2\pi} -(1 - u^2) du$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right) d\theta + 2r \left[u - \frac{1}{3}u^3\right]_{\theta=0}^{2\pi}$$

$$= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\theta=0}^{2\pi} + 2r \left[\cos \theta - \frac{1}{3}(\cos^3 \theta)\right]_{\theta=0}^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0) - \frac{1}{4}(\sin(4\pi) - \sin(0)) + 2r \left(\cos(2\pi) - \cos(0)\right) - \frac{1}{3}(\cos^3(2\pi) - \cos^3(0))$$

$$= \pi - \frac{1}{4} \cdot 0 + 2r(0 - \frac{1}{3} \cdot 0) = \pi$$

eval ukr integral:

$$\int_{r=0}^1 2\sqrt{2} r^3 \pi dr = \frac{2\sqrt{2} \pi}{4} [r^4] \Big|_{r=0}^1$$

$$\therefore \frac{\pi}{\sqrt{2}} (1^4 - 0^4) = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_C 4^u dx + 2xy^2 dy = \boxed{\frac{\pi}{\sqrt{2}}}$$

NB: So far, all examples so far have turned line integrals into double integrals via Green's Theorem. BUT we can go in other way as well.

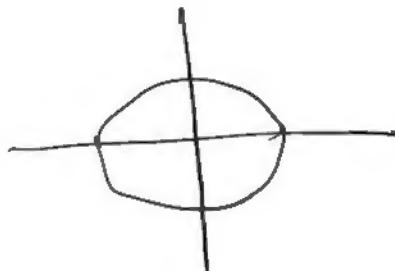
NB: If P, Q satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,

then via Green's theorem

$$\int_D P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D 1 dA = \text{Area}(D)$$

Ex: Compute the area of the general ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Soln: $\text{Area}(D) = \int_D P dx + Q dy$

if we choose

$$P, Q \text{ w/ } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

e.g. $Q=0, P=-y: \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}[0] - \frac{\partial}{\partial y}[-y] = 1$

$$\therefore \text{area}(D) = \int_D -y dx + 0 dy = - \int_D y dx$$

find 2D:

the ellipse 2D is parameterized by

$$\vec{r}(t) = \langle a \cos(t), b \sin(t) \rangle \text{ on}$$

$$0 \leq t \leq 2\pi \quad \therefore dx = x'(t) dt$$

$$= - \int_{t=0}^{2\pi} b \sin(t) \cdot -a \sin(t) dt$$

$$= ab \int_{t=0}^{2\pi} \sin^2(t) dt$$

$$= ab \int_{t=0}^{2\pi} \frac{1}{2}(1 - \cos(2t)) dt$$

$$= \frac{1}{2} ab \left[t - \frac{1}{2} \sin(2t) \right]_{t=0}^{2\pi}$$

$$= \frac{1}{2} ab \left((2\pi - 0) - \frac{1}{2}(0 - 0) \right) = \boxed{ab\pi}$$

§ 16.5: Curl and Divergence

Goal: Define and Study two new operations on vector fields.

Curl: The Curl of vector field \vec{v} on \mathbb{R}^3 is
 $\vec{v} = (P, Q, R)$

$$\text{Curl}(\vec{v}) = " \nabla \times \vec{v} " = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times (P, Q, R)$$

$$= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Ex: Compute $\text{curl}(\vec{v})$ for $\vec{v} = \langle xy, xyz, -y^2 \rangle$

Sol:

$$\text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xyz & -y^2 \end{bmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}[-y^2] - \frac{\partial}{\partial z}[xyz], -\left[\frac{\partial}{\partial x}[-y^2] - \frac{\partial}{\partial z}[xy] \right], -\frac{\partial}{\partial y}[xy] \right\rangle$$

$$= \langle -2y - xy, -(0-0), yz - x \rangle$$

$$= \langle -xy - 2y, 0, yz - x \rangle$$

Observation: Suppose $\vec{v} = \nabla f$ is conservative.

i.e. $\vec{v} = \langle f_x, f_y, f_z \rangle$

$$\text{So, } \text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{bmatrix}$$

$$= \left\langle \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, -\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z} \right), \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right\rangle$$

$$\begin{aligned}
 &= \langle \hat{f}_{2y} - \hat{f}_{y2}, -(\hat{f}_{2x} - \hat{f}_{x2}), \hat{f}_{yx} - \hat{f}_{xy} \rangle \\
 &= \langle 0, 0, 0 \rangle = \vec{0} \text{ by Clairaut's theorem.}
 \end{aligned}$$

Point: $\text{Curl}(\vec{0}) = \vec{0}$, i.e. Curl of a conservative \vec{f} is $\vec{0}$.

Prop: A vector field w/ components having cts. partial derivatives is conservative if and only if $\text{Curl}(\vec{v}) = \vec{0}$.

Divergence: The divergence of vector field \vec{v} , $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$ is

$$\begin{aligned}
 \text{div}(\vec{v}) = " \vec{v} \cdot \vec{i} " &= \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right\rangle \cdot \langle v_1, v_2, \dots, v_n \rangle \\
 &= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \dots + \frac{\partial v_n}{\partial x_n}
 \end{aligned}$$

Ex: For $\vec{v} = \langle xy, xyz, -y^2 \rangle$ compute
 $\operatorname{div}(\vec{v})$.

$$\begin{aligned} \text{Sol. } \operatorname{div}(\vec{v}) &= \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[xyz] + \frac{\partial}{\partial z}[-y^2] \\ &= 1 + xz + 0 = 1 + xz \end{aligned}$$

Suppose: $\vec{v} = \operatorname{curl}(\vec{\omega})$ for $\omega = \langle P, Q, R \rangle$

$$\vec{v} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$$

$$\begin{aligned} \text{Now, } \operatorname{div}(\vec{v}) &= \frac{\partial}{\partial x}[R_y - Q_z] + \frac{\partial}{\partial y}[-(R_x - P_z)] + \frac{\partial}{\partial z}[Q_x - P_y] \\ &= (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) + (R_{yx} - R_{xy}) \end{aligned}$$

$$\begin{aligned} &= (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) + (R_{yx} - R_{xy}) \\ &= 0 + 0 + 0 = 0 \quad \text{by Clairaut's theorem.} \end{aligned}$$

Prop: A vector field is the curl of another vector field if and only if its divergence is zero.

NB: above we should $\operatorname{div}(\operatorname{curl}(\vec{w}))=0$.